Name: ______________________ GTID: ______________________

- Fill out your name and Georgia Tech ID number.
- This exam contains 13 pages. Please make sure no page is missing.
- The grading will be done on the scanned images of your test. Please write clearly and legibly.
- Answer the questions in the spaces provided. Please try to fit your solutions into the solution boxes. **We will scan the front sides only by default.** If you run out of room for an answer, continue on the back of the page, indicate here (below) that you did so, and notify the proctor when handing in, in order to receive the due credit for your work:

  ______________________________________________

- Please write detailed solutions including all steps and computations. Include worded explanations when necessary. Do not just state answers without explanations.
- The duration of the exam is 170 minutes.
- **There is a formulae sheet at the back.**

Good luck!

1. (12 points) Verify that $y_1(t) = e^{-4t}$ and $y_2(t) = e^{4t}$ form a fundamental set of solutions of $y'' - 16y = 0$.

**Solution:**

For $y_1 = e^{-4t}$, $y_1^{'} = -4e^{-4t}$ and $y_1^{''} = 16e^{-4t}$. So,

$$y_1^{''} - 16y_1 = 0. \longrightarrow \text{4 points}$$

For $y_2 = e^{4t}$, $y_2^{'} = 4e^{4t}$ and $y_2^{''} = 16e^{4t}$. So,

$$y_2^{''} - 16y_2 = 0. \longrightarrow \text{4 points}$$

$W(y_1, y_2) = \begin{vmatrix} e^{-4t} & e^{4t} \\ -4e^{-4t} & 4e^{4t} \end{vmatrix} = 8 \neq 0. \longrightarrow$ 3 points

Hence, $y_1(t)$ and $y_2(t)$ form a fundamental set of solutions of $y'' - 16y = 0. \longrightarrow$ 1 point

2. (10 points) Find the general solution to $\frac{dy}{dt} - y = e^{3t}$. Write down the solution in an explicit form.

**Solution:** Integrating factor is $e^{-\int dt} = e^{-t}$. $\longrightarrow$ 2 points

Multiplying both sides of the DE by the integrating factor and rearranging the terms, we get

$\frac{d}{dt}(ye^{-t}) = e^{2t}$ $\longrightarrow$ 3 points

Integrating both sides we get

$ye^{-t} = \frac{1}{2}e^{2t} + c.$ $\longrightarrow$ 4 points

Rearranging the terms, we get

$y = \frac{1}{2}e^{3t} + ce^{t}.$ $\longrightarrow$ 1 point

3. (12 points) a) Find the general solution of

$$\mathbf{X}' = \begin{pmatrix} 1 & 5 \\ -2 & -1 \end{pmatrix} \mathbf{X} + \begin{pmatrix} -6 \\ 3 \end{pmatrix}.$$

Express the answer using real-valued functions.

b) Choose the correct statement below: The critical point of the above system is

◯ asymptotically stable, $\surd$ **stable (but not asymptotically stable) or** ◯ unstable

**Solution:** a) We first take an appropriate change of variable to make the system homogeneous. In particular we take $\mathbf{U} = \mathbf{X} + \begin{pmatrix} a \\ b \end{pmatrix}$ where

$$\begin{pmatrix} 1 & 5 \\ -2 & -1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -6 \\ 3 \end{pmatrix}.$$

Solving the corresponding linear system yields $a = b = -1$. $\longrightarrow$ 3 points

a) Eigenvalues: $\mid \mathbf{A} - \lambda \mathbf{I} \mid = \begin{vmatrix} 1-\lambda & 5 \\ -1 & -1-\lambda \end{vmatrix} = \mathbf{0}$

$\Longleftrightarrow$ $\lambda_1 = 3i, \lambda_2 = -3i \longrightarrow$ 3 points

Eigenvectors for $\lambda_1 = 3i : (\mathbf{A} - (3i)\mathbf{I})\mathbf{X} = \mathbf{0} \iff$

$$\begin{pmatrix} 1-3i & 5 \\ -2 & -1-3i \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff v_2 = -\frac{1-3i}{5} v_1$$

$$\text{When } v_1 = 5, \text{ the eigen vector } \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 5 \\ -1+3i \end{pmatrix} \longrightarrow \text{3 points}$$

Note that $\text{Re}(\mathbf{v}) = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$ and $\text{Im}(\mathbf{v}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

So, the general solution is :

$$\mathbf{U} = c_1 \left( \begin{pmatrix} 5 \\ -1 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t \right) + c_2 \left( \begin{pmatrix} 5 \\ -1 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos 3t \right) \text{ or}$$

$$\iff \mathbf{X} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_1 \begin{pmatrix} 5\cos 3t \\ -\cos 3t - 3\sin 3t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin 3t \\ 3\cos 3t - \sin 3t \end{pmatrix} \longrightarrow \text{2 points}$$

**Note: The general solution may vary**

b) Since roots are complex with real part equal to 0 we conclude that the critical point is stable, but not asymptotically stable.$\longrightarrow$ 1 points

4. (10 points) Find explicitly the general solution of the differential equation

$$y'' + 3y' + 2y = \cos(t).$$

**Solution:** The associated characteristic equation is

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2), \longrightarrow \text{1 points}$$

from which we conclude that the homogeneous solution is

$$y_c(t) = c_1 e^{-t} + c_2 e^{-2t}. \longrightarrow \text{2 points}$$

To compute the particular solution we use the method of undetermined coefficients and assume that $y_p(t) = A\cos(t) + B\sin(t)$. In particular it satisfies

$$\begin{aligned}\cos(t) &= -A\cos(t) - B\sin(t) + 3(-A\sin(t) + B\cos(t)) + 2(A\cos(t) + B\sin(t)) \\ &= (3B + A)\cos(t) + (-3A + B)\sin(t) \longrightarrow \text{3 points}\end{aligned}$$

Therefore $A, B$ satisfy $3B + A = 1$ and $-3A + B = 0$, from which we get $A = 1/10, B = 3/10$. $\longrightarrow$ 2 points Therefore the particular solution is

$$y_p(t) = \frac{\cos(t)}{10} + \frac{3\sin(t)}{10} \longrightarrow \text{1 points}$$

Using the fact that the general solution is $y_p + y_c$ and rewriting the solution in terms of $x$, we conclude that the general solution is

$$y(x) = c_1 e^{-t} + c_2 e^{-2t} + \frac{\cos(t)}{10} + \frac{3\sin(t)}{10}. \longrightarrow \text{1 points}$$

5. (12 points) Find the particular solution of $y'' + 3y' + 2y = te^{-t}$. Note that $y_c(t) = c_1 e^{-t} + c_2 e^{-2t}$. Use the method of variation of parameters.

**Solution:** Fundamental matrix: $\Phi(t) = \begin{pmatrix} e^{-t} & e^{-2t} \\ -e^{-t} & -2e^{-2t} \end{pmatrix} \to$ 1 points

$\mid \Phi(t) \mid = -e^{-3t} \to$ 1 point

$\Phi^{-1}(t) = \begin{pmatrix} 2e^{t} & e^{t} \\ -e^{2t} & -e^{2t} \end{pmatrix} \to$ 2 points

$\begin{pmatrix} u_1'(t) \\ u_2'(t) \end{pmatrix} = \Phi^{-1}(t)\ G(t) = \begin{pmatrix} t \\ -te^{t} \end{pmatrix} \to$ 2 points

So, $\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} t^2/2 \to \text{2 points} \\ (1-t)e^{t} \to \text{2 points} \end{pmatrix}$

So,

$$\begin{aligned} y_p(t) &= u_1(t)e^{-t} + u_2(t)te^{-2t} \\ &= \frac{1}{2}t^2 e^{-t} + (1-t)e^{-t} \to \text{2 points} \end{aligned}$$

6. (10 points) Let $g : [0, \infty) \to \mathbb{R}$ be a function. Find the solution to the following initial value problem:

$$y'' + 4y' + 4y = g(t), y(0) = 2, y'(0) = -3$$

Use a convolution integral to express the particular solution in dependence of $g$.

**Solution:** We have

$$\mathcal{L}(y'' + 4y' + 4y) = \mathcal{L}(g(t)) \longrightarrow \text{2 points}$$

$$\implies s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 4Y(s) = G(s) \longrightarrow \text{2 points}$$

$$\implies Y(s)(s^2 + 4s + 4) = (2s + 5) + G(s) \longrightarrow \text{1 points}$$

$$\implies Y(s) = \frac{2s + 5}{s^2 + 4s + 4} + \frac{G(s)}{s^2 + 4s + 4} \longrightarrow \text{1 points}$$

$$\implies Y(s) = \frac{2}{s + 2} + \frac{1}{(s + 2)^2} + \frac{G(s)}{(s + 2)^2} \longrightarrow \text{2 points}$$

$$\implies y(t) = (t + 2)e^{-2t} + \int_0^t g(\tau)e^{-2(t-\tau)}(t - \tau)\, d\tau \longrightarrow \text{2 points}$$

7. (8 points) Find the critical points of the linear system

$$\begin{cases} \frac{dx}{dt} = x(3 - x - 4y) \\ \frac{dy}{dt} = y(1 - 3x). \end{cases}$$

**Solution:**

The critical points are found by solving the equations

$$x(3 - x - 4y) = 0 \text{ and } y(1 - 3x) = 0.$$

Satisfy the first equation by choosing $x = 0$ or $3 - x - 4y = 0$. $\longrightarrow$ 2 points

Substituting the $x = 0$ into the second equation yields the critical point $(0, 0) \rightarrow$ 2 points.

Substituting $3 - x - 4y = 0$ into the second equation yields the critical points $(3, 0) \rightarrow$ 2 points and $\left(\frac{1}{3}, \frac{2}{3}\right) \rightarrow$ 2 points.

8. (8 points) a) Write the corresponding linear system of

$$\begin{cases} \frac{dx}{dt} = x(3-x-4y) \\ \frac{dy}{dt} = y(1-3x). \end{cases}$$

at all the critical points (which you found in question 7).

**Solution:** Note that

$$F(x,y) = x(3-x-4y) \text{ and } G(x,y) = y(1-3x).$$

The Jacobian matrix for the system is

$$\begin{pmatrix} F_x & F_y \\ G_x & G_y \end{pmatrix} = \begin{pmatrix} 3-2x-4y & -4x \\ -3y & 1-3x \end{pmatrix}. \rightarrow \text{2 points}$$

Thus the corresponding linear system at $(3,0)$ is $\begin{pmatrix} u' \\ w' \end{pmatrix} = \begin{pmatrix} -3 & -12 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \rightarrow$ 2 points where $u = x-3$ and $w = y-0 = y$.

Next, the corresponding linear system at $(0,0)$ is

$$\begin{pmatrix} u' \\ w' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \rightarrow \text{2 points}$$

where $u = x$ and $w = y$.

Lastly, the corresponding linear system at $(1/3, 2/3)$ is

$$\begin{pmatrix} u' \\ w' \end{pmatrix} = \begin{pmatrix} -1/3 & -4/3 \\ -2 & 0 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \rightarrow \text{2 points}$$

where $u = x - 1/3$ and $w = y - 2/3$.

9. (6 points) Determine if each of the critical points of

$$\begin{cases} \frac{dx}{dt} = x(3 - x - 4y) \\ \frac{dy}{dt} = y(1 - 3x). \end{cases}$$

(found in question 7) is asymptotically stable, stable (but not asymptotically stable), unstable or unclear using this method. Make sure to include all the explanations.

**Solution:** The eigenvalues of $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ are $3, 1$. As both of the eigenvalues are positive, we get that the equilibrium point $(0, 0)$ is unstable. $\rightarrow$ 2 points

The eigenvalues of $\begin{pmatrix} -1/3 & -4/3 \\ -2 & 0 \end{pmatrix}$ are roots of the equation

$$(-1/3 - \lambda)(-\lambda) - 8/3 = 0,$$

of them one positive and one negative, so we get that the equilibrium point $(1/3, 2/3)$ is unstable. $\rightarrow$ 2 points

The eigenvalues of $\begin{pmatrix} -3 & -12 \\ 0 & -8 \end{pmatrix}$ are roots of the equation

$$(-3 - \lambda)(-8 - \lambda) = 0,$$

which are $-3, -8$. As both of the eigenvalues are negative, we get that the equilibrium point $(3, 0)$ is an asymptotically stable node of the nonlinear system. $\rightarrow$ 2 points

10. (12 points) Consider

$$\frac{dy}{dt} = t + y^2, \ y(0) = 0.$$

For this differential equation, write down the general formula for Euler's method (for evaluating $y_{n+1}$ in terms of $y_n$ and $t_n$) and find the approximate value at $t = 0.2$ using the explicit Euler method with the step size $h = 0.1$

**Solution:** $f(t, y) = t + y^2$

$t_0 = 0$, $y_0 = 0$.

Note $y_{n+1} = y_n + h(t_n + y_n^2) \rightarrow$ 4 points.

$y_1 = y_0 + (0.1) * (t_0 + y_0^2) = 0 \rightarrow$ 4 points.

$y_2 = y_1 + (0.1) * (t_1 + y_1^2) = 0.01 \rightarrow$ 4 points.

11. Bonus (10 points) Find a solution to the ODE

$$f(x) = \sum_{n=1}^{\infty} f^{(n)}(x)$$

with $f(0) = 1$.
Hint: Use the Laplace transform on the ODE and try to use the identity

$$1 + s + s^2 + s^3 + \cdots = \frac{1}{1-s}.$$

You will need to play around with the resulting series a bit :)

**Solution:**

$$f(x) = e^{\frac{x}{2}}.$$

To see this, we will solve the ODE using the Laplace transform. We have

$$\mathcal{L}(f) = \mathcal{L}\left(\sum_{n=1}^{\infty} f^{(n)}\right) = \sum_{n=1}^{\infty} \mathcal{L}\left(f^{(n)}\right)$$

$$\implies F(s) = \sum_{n=1}^{\infty} \left(s^n F(s) - s^{n-1} f^{(0)}(0) - \cdots - f^{(n-1)}(0)\right).$$

Rearranging terms one has

$$F(s)(1 - s - s^2 - \cdots) = -\sum_{k=0}^{\infty} (f^k(0) + s f^k(0) + s^2 f^k(0) + \cdots).$$

Using the fact that $1+s+s^2+\cdots = \frac{1}{1-s}$ and the fact that $f^{(0)}(0)+f^{(1)}(0)+f^{(2)}(0)+\cdots = 2f(0) = 2$, the right hand side is equivalent to $-\frac{2}{1-s}$. Also, by adding and subtracting 2 copies of $F(s)$ from the left hand, we may write the left hand side as $F(s)\left(2 - \frac{1}{1-s}\right)$. Therefore we have

$$F(s)\left(2 - \frac{1}{1-s}\right) = -\frac{2}{1-s} \implies F(s)\left(\frac{1-2s}{1-s}\right) = -\frac{2}{1-s} \implies F(s) = -\frac{2}{1-2s} = \frac{1}{s-1/2}.$$

Therefore

$$f(x) = \mathcal{L}^{-1}(F) = \mathcal{L}^{-1}\left(\frac{1}{s-1/2}\right) = e^{x/2}.$$

Some important formulae

- $\mathcal{L}\{1\} = \dfrac{1}{s}$, $s > 0$.
- $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$, $s > 0$, $n =$ positive integer.
- $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$, $s > a$.
- $\mathcal{L}\{\sin kt\} = \dfrac{k}{s^2+k^2}.$ , $s > 0$
- $\mathcal{L}\{\cos kt\} = \dfrac{s}{s^2+k^2}$, $s > 0$.
- $\mathcal{L}\{\sinh kt\} = \dfrac{k}{s^2-k^2}$, $s >| k |$.
- $\mathcal{L}\{\cosh kt\} = \dfrac{s}{s^2-k^2}$, , $s >| k |$.
- $\mathcal{L}\{u_c(t)\} = \dfrac{e^{-cs}}{s}$, $s > 0$
- $\mathcal{L}\Big\{u_c(t)f(t-c)\Big\} = e^{-cs}F(s)$.
- $\mathcal{L}\{e^{ct}\ f(t)\} = F(s-c), s > a + c$.
- $\mathcal{L}\{t^n f(t)\} = (-1)^n F^{(n)}(s), s > a$.
- $\mathcal{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)$
- $(f \star g)(t) = \displaystyle\int_0^t f(t-\tau)g(\tau)d\tau$
- $\mathcal{L}\Big\{f \star g\Big\} = F(s)\ G(s)$